A GENERAL THEORY OF LIMITS.

By E. H. Moore and H. L. Smith.

Introduction.

I. *As to Notions.*—In this paper we investigate a simple general limit of which, as will appear in § 5, the various classical limits of analysis are actually instances.* The general limit in question is an obvious generalization of the following two limits:

1. An infinite sequence $\{a_n\}$ of real or (ordinary) complex numbers a_n $(n = 1, 2, \cdots)$ converges to a number a as a limit, in notation: $L_{n\to\infty} a_n = a$,—as clearly defined about a century ago,—in case for every positive number e there exists a positive integer n_e of such a nature that for every integer $n \geqq n_e$ it is true that in absolute value $a_n - a$ is at most e. Here the numerical sequence $\{a_n\}$ may be considered as a numerically valued function $\alpha \equiv (a_n | n)$ of the positive integer n (or on the range $[n]$ of positive integers n), viz., $\alpha(n) \equiv a_n$ for every n.

2. Relative to a general (i.e., any particular) class $\mathfrak{Q} \equiv [q]$ of general elements q and the class $\mathfrak{S} = [s]$ of all finite classes s of elements q, a numerically valued function $\alpha \equiv (\alpha(s) | s)$ on the range S converges to a number a as limit, in notation: $L_s\alpha(s) = a$, in case for every positive number e there exists a class s_e of such a nature that for every class s including s_e it is true that in absolute value $\alpha(s) - a$ is at most e. The limit (2) belongs to General Analysis, i.e., to that doctrine of analysis in which a general class, here $\mathfrak{Q}$, plays a fundamental rôle. The limit (2), introduced† in 1915 by the senior author, plays a central rôle in his second theory‡ of Linear Integral Equations in General Analysis. This general theory has as notable instances: (*a*) Hilbert's theory of limited quadratic forms in a denumerable infinitude of variables; here the Hilbert space $[\alpha]$ of infinite sequences $\alpha \equiv (\alpha(n) | n)$ of real numbers with convergent $\Sigma_n\alpha(n)\alpha(n)$ plays a central rôle; (*b*) an analogous theory in which the corresponding rôle

* The only other attempt in this direction is by Dimitry-Kryjanowsky, *Nouvelles Annales de Mathématiques*, Ser. 4, Vol. 14 (1914), pp. 49–64. In this paper it is shown how all the classical limits, by means of a transformation, may be reduced to a certain canonical form. The theory is not as general as the present one; indeed it does not include the limit 2) below.

† E. H. Moore, "Definition of Limit in General Integral Analysis," *Proceedings of the National Academy of Sciences*, Vol. 1 (1915), pp. 628–632.

‡ (Addition of Sept. 19, 1922.) The basis of this theory is included in his paper, "On Power Series in General Analysis," *Festschrift David Hilbert zu seinem sechzigsten Geburtstag*, pp. 355–364, Berlin, 1922, and *Mathematische Annalen*, vol. 86, pp. 30–39 (1922).

is played by the Hellinger space $[\alpha]$ of real-valued functions $\alpha \equiv (\alpha(x)\,|\,x)$ of the real variable x on the interval $(0 \leqq x \leqq 1)$ with $\alpha(0) = 0$ and $\int_0^1 \frac{d\alpha(x)d\alpha(x)}{d\beta(x)}$ convergent, β being a fixed monotone increasing function of x; (c) various instances involving integration over function-spaces.

It is obvious that the limits (1), (2) are in close analogy. In each case a numerically valued function $\alpha = (\alpha(p)\,|\,p)$ on a certain range $\mathfrak{P} \equiv [p]$ is said to converge, under certain conditions, to a number a as limit; in (1) $\mathfrak{P} \equiv [n]$, in (2) $\mathfrak{P} \equiv \mathfrak{S} \equiv [s]$; moreover, notations apart, the conditions in (1) become the conditions in (2) on replacing the relation $\geqq$, between two positive integers ($n \geqq n_e$), by the relation *including* or $\supset$, between two classes ($s \supset s_e$). Thus the authors were led independently to the general limit (3) of which the limits (1), (2) are special instances.

3. Consider a *general class* $\mathfrak{P} \equiv [p]$ of members or *elements* p and a *binary relation* R on the class $\mathfrak{P}$; according as an element p_1 is or is not in the relation R to an element p_2 write $p_1 \mathrm{R} p_2$ or $p_1 {}^{-}\mathrm{R} p_2$. Then we say that a *numerically valued function* $\alpha \equiv (\alpha(p)\,|\,p)$* *on the range* $\mathfrak{P}$ *converges* (*with respect to the relation* R) *to a number* a *as limit,* in notation:

$$\mathrm{L}_p \alpha(p) = a \qquad \text{or} \qquad \mathrm{L}\alpha = a,$$

or, with the relation R in evidence,

$$\mathrm{L}_{p\mathrm{R}} \alpha(p) = a \qquad \text{or} \qquad \mathrm{L}_{\mathrm{R}}\alpha = a,$$

in case for every positive number e *there exists an element* p_e *of such a nature that for every* $p\mathrm{R}p_e$ (i.e., for every element p in the R relation to p_e) *it is true that in absolute value* $\alpha(p) - a$ *is at most* e.†

In order that the theory of the general limit (3) may include the principal parts of the theories of the limits (1), (2), we impose upon the relation R two conditions obviously satisfied in (1), (2), viz., the conditions: 1) R is *transitive* (R^T),—if p_1 is in the R relation to p_2 and p_2 is in the R relation to p_3, then p_1 is in the R relation to p_3; 2) R has the *composition property* (R^C),—for every two (not necessarily distinct) elements $p_1 p_2$ there exists an element $p_3 \mathrm{R}(p_1, p_2)$, that is, an element p_3 in the R relation to each of the elements $p_1 p_2$.

Thus, as *fundamental system* Σ *of notions* for the general limit (3), we have the system:

$$\Sigma \equiv (\mathfrak{A};\ \mathfrak{P};\ \mathrm{R}^{\text{on}\,\mathfrak{P}\mathfrak{P}\cdot TC});$$

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* Throughout the paper α will denote a numerically valued function on the range, not restricted to be single-valued unless so stated.

† If α is not single-valued, this inequality is understood to hold for all determinations of $\alpha(p)$; unless otherwise stated similar understandings will hold in the case of all inequalities involving multiply-valued functions.

that is, the class $\mathfrak{A} \equiv [a]$ of all real or (ordinary) complex numbers a, a general class $\mathfrak{P}$, and a binary relation R on the class $\mathfrak{P}$ which is transitive and has the composition property.

II. *As to Scope.*—The present study of the general limit (3) is arranged as follows:

§ 1. Elementary theorems.
§ 2. Necessary and sufficient conditions for the existence of a limit.
§ 3. Some modes of convergence.
§ 4. Quasi-limits. Upper and lower limits.
§ 5. Limits as to norm.
§ 6. Types of uniform convergence as to a general parameter.
§ 7. Double limits.
§ 8. Lemmas: Revised formulation of certain theorems of Fréchet.
§ 9. Composite range. Continuity.

III. *As to Notations.*—In order to expound briefly and luminously the considerable body of doctrine outlined in II, we make systematic use of readily understood notations for constantly recurring logical and mathematical notions.

For instance, the definition in I (3) of $L\alpha = a$ we write: *there exists a system* $(p_e | e)$ *such that for every* $p\mathrm{R}p_e$ *it is true that* $|\alpha(p) - a| \leqq e$, or even more briefly, *there exists a system* $(p_e|e)$ *such that*

$$|\alpha(p) - a| \leqq e \qquad (p\mathrm{R}p_e).$$

Throughout e denotes a positive number.

§ 1. Elementary Theorems.

In the Introduction we have indicated the fundamental system

$$\Sigma = (\mathfrak{A};\ \mathfrak{P};\ \mathrm{R}^{\mathrm{on}\,\mathfrak{P}\mathfrak{P}\cdot TC})$$

of notions under consideration, and defined the associated limit notion. We here add certain simple explanations and propositions.

It follows from R^C that for every element p there is an element p_p such that $p_p\mathrm{R}p$. For we have only to take in the definition of R^C $p_1 = p$, $p_2 = p$ and then take $p_p = p_3$.

In case $p\mathrm{R}p$ for every p, R is *reflexive,* in notation R^R. It is not assumed that R is reflexive. We shall however define an associated relation R_* which is transitive, has the composition property and is also reflexive. We define: $p_1\mathrm{R}_*p_2$ in case either $p_1\mathrm{R}p_2$ or $p_1 = p_2$. The proof that ${\mathrm{R}_*}^{TCR}$ is simple and is omitted.

To the definition of limit given in the introduction should be added the following one: α *converges to* $\sigma\infty$ ($\sigma = +$ or $-$) *as limit,* in notation:

$$L_p\alpha(p) = \sigma\infty \qquad \text{or} \qquad L\alpha = \sigma\infty,$$

or, with the relation R in evidence,

$$L_{p\text{R}}\alpha(p) = \sigma\infty \qquad \text{or} \qquad L_{\text{R}}\alpha = \sigma\infty$$

in case there exists a system $(p_e|e)$ *such that*

$$\sigma\alpha(p) \geqq e \qquad (p\text{R}p_e).$$

The limit of α *exists* in case there exists a number a such that $L\alpha = a$. The limit of α *exists finite or infinite* in case either $L\alpha = +\infty$, $L\alpha = -\infty$, or there is some a such that $L\alpha = a$.

By $\alpha(\mathfrak{P}_0)$ will be denoted the function α *reduced to be on* $\mathfrak{P}_0 = [p_0]$, a subclass of $\mathfrak{P}$; that is, α considered only for elements p_0 of $\mathfrak{P}_0$. Every reduced function $\alpha(\mathfrak{P}_0)$ gives rise to a reduced limit $L\alpha(\mathfrak{P}_0)$, if R as on $\mathfrak{P}_0\mathfrak{P}_0$ has the property C.* The definition of $L\alpha(\mathfrak{P}_0)$ is the same as that of $L\alpha$, except that all elements p must now be restricted to be of $\mathfrak{P}_0$.

By $\mathfrak{A}^*$ will be denoted $\mathfrak{A}$ enlarged by the addition of $+\infty$ and $-\infty$. Any element of $\mathfrak{A}^*$ will be denoted by a^*.

0. If $L_{\text{R}}\alpha = a^*$, then $L_{\text{R}_*}\alpha = a^*$, and conversely.

1. *If* $L\alpha = a_1^*$ *and* $L\alpha = a_2^*$, *then* $a_1^* = a_2^*$.

By proper choice of notation the possible cases may be reduced to six:

1) a_1^* and a_2^* both finite.

2σ) $a_1^* = \sigma\infty$, $\quad a_2^* = \sigma\infty$.

3σ) $a_1^* = \sigma\infty$, $\quad a_2^*$ finite.

4) $a_1^* = +\infty$, $\quad a_2^* = -\infty$.

Of these 3σ and 4 lead to contradictions, leaving only (1) to be considered since 2σ are in harmony with the theorem. Let us consider (1).

By hypothesis there are systems $(p_{1e}|e)$, $(p_{2e}|e)$ such that

$$|\alpha(p) - a_1^*| \leqq \frac{e}{2} \qquad (p\text{R}p_{1e}),$$

$$|\alpha(p) - a_2^*| \leqq \frac{e}{2} \qquad (p\text{R}p_{2e}).$$

By R^C there then exists a system $(p_e|e)$ such that $(p_e\text{R}p_{1e},\ p_e\text{R}p_{2e})$ for every e. Then

$$|a_1^* - a_2^*| \leqq |a_1^* - \alpha(p_e)| + |\alpha(p_e) - a_2^*| \leqq e$$

for every e, so that $|a_1^* - a_2^*| = 0$ and $a_1^* = a_2^*$.

2. *If* $L\alpha = a$, *then* $L|\alpha| = |a|$.†

3. *If* $L\alpha = a$, *then* $L(c\alpha) = ca \quad (c)$.

4. *If* $L\alpha_1 = a_1$, $L\alpha_2 = a_2$, *then*

$$L(\alpha_1 + \alpha_2) = a_1 + a_2,$$
$$L(\alpha_1\alpha_2) = a_1a_2,$$
$$L\frac{\alpha_1}{\alpha_2} = \frac{a_1}{a_2} \qquad (a_2 \neq 0).$$

* The relation R as on $\mathfrak{P}_0\mathfrak{P}_0$ is necessarily T.

† Here $|\alpha|$, *the absolute of* α, denotes the function: $|\alpha|(p) = |\alpha(p)| \qquad (p)$.

5. *If* Lα *exists and, for certain* cdp_0, $|\alpha(p) - c| \leqq d$ *for* $p\text{R}p_0$, *then* $|\text{L}\alpha - c| \leqq d$.

6. *If* $\alpha(p) \geqq c$ *for every* p *and* Lα *exists, then* L$\alpha \geqq c$.

The proofs of propositions 2–6 are simple.

A single-valued function α is *monotone increasing* (*relative to* R) if $\alpha(p_1) \geqq \alpha(p_2)$ for every pair (p_1, p_2) such that $p_1\text{R}p_2$; *properly* so if $\alpha(p_1) > \alpha(p_2)$ for every pair of distinct elements (p_1, p_2) such that $p_1\text{R}p_2$. The term *monotone decreasing* is similarly defined.

7. *If* α *is single-valued and monotone, then* Lα *exists finite or infinite and is equal to* $\overline{B}\alpha$* *or* $\underline{B}\alpha$, *that is,* $\overline{B}_p\alpha(p)$ *or* $\underline{B}_p\alpha(p)$, *according as* α *is monotone increasing or monotone decreasing.*

Assume first that α is monotone increasing and that $\overline{B}_p\alpha(p) = a$, a finite number. Then there exists a system $(p_e|e)$ such that

$$a \geqq \alpha(p_e) \geqq a - e \qquad (e).$$

Then

$$a \geqq \alpha(p) \geqq \alpha(p_e) \geqq a - e \qquad (p\text{R}p_e) \qquad (e);$$

and hence,

$$|\alpha(p) - a| \leqq e \qquad (p\text{R}p_e) \qquad (e);$$

so that L$\alpha = a$. The other cases are treated similarly.

§ 2. Necessary and Sufficient Conditions for the Existence of a Limit.

1. (Cauchy Condition). *In order that* Lα *shall exist it is necessary and sufficient that there shall exist a system* $(p_e|e)$ *such that*

$$|\alpha(p_1) - \alpha(p_2)| \leqq e \qquad (p_1\text{R}p_e,\ p_2\text{R}p_e).$$

This condition is *necessary.* For there exists a system $(p_e|e)$ such that

$$|\text{L}\alpha - \alpha(p)| \leqq \frac{e}{2} \qquad (p\text{R}p_e) \qquad (e).$$

The condition is also *sufficient.* For there exists a sequence $\{p_n\}$ such that $p_{n+1}\text{R}p_n$ $(n = 1, 2, 3, \cdots)$ and

$$|\alpha(p') - \alpha(p'')| \leqq \frac{1}{2^n}$$

for $(p'\text{R}p_n,\ p''\text{R}p_n)$. The numbers $\alpha(p_1), \alpha(p_2), \cdots$ form a limited or bounded set since

$$|\alpha(p_n) - \alpha(p_2)| \leqq |\alpha(p_n) - \alpha(p_{n-1})| + \cdots + |\alpha(p_3) - \alpha(p_2)|$$
$$\leqq \frac{1}{2^{n-2}} + \cdots + \frac{1}{2^1} \leqq 1,$$

for every $n > 2$. Hence there exists a subsequence $\{p_{n_m}\}$ such that the

* Read: *the* (*least*) *upper bound of* α.

numerical sequence $\{\alpha(p_{n_m})|m\}$ approaches some number a as a limit.* That $\mathrm{L}\alpha = a$ now follows from the inequality

$$|\alpha(p) - a| \leqq |\alpha(p) - \alpha(p_{n_m})| + |\alpha(p_{n_m}) - a|$$
$$\leqq \frac{1}{2^{n_m}} + |\alpha(p_{n_m}) - a|,$$

which holds for $p\mathrm{R}p_{n_m}$.

The above theorem is true if R is replaced by R_* in the final line, as is evident by § 1, 0.

2. *In order that* $\mathrm{L}\alpha$ *shall exist it is necessary and sufficient that*

$$\mathrm{L}_{p_1p_2}[\alpha(p_1) - \alpha(p_2)] = 0,$$

that is, that there exist systems $(p_{1e}|e)$, $(p_{2e}|e)$ such that

$$|\alpha(p_1) - \alpha(p_2)| \leqq e \qquad (p_1\mathrm{R}p_{1e},\ p_2\mathrm{R}p_{2e}) \qquad (e).$$

This theorem follows at once from the preceding and the composition property of R.

3. *In order that* $\mathrm{L}\alpha$ *shall exist it is necessary and sufficient that there exist a system* $(p_e|e)$ *such that*

$$|\alpha(p) - \alpha(p_e)| \leqq e \qquad (p\mathrm{R}p_e) \qquad (e).$$

The necessity of this condition follows from the R_* form of theorem 1 and its sufficiency also on taking the p_e of the required $(p_e|e)$ of that theorem as the given $p_{e/2}$ of the $(p_e|e)$ of the present theorem.

4. *In order that* $\mathrm{L}\alpha$ *shall not exist it is necessary and sufficient that there exist* e_0 *and a system* $(p_{1p}, p_{2p}|p)$ *such that*

$$p_{1p}\mathrm{R}p,\ p_{2p}\mathrm{R}p,\ |\alpha(p_{1p}) - \alpha(p_{2p})| > e_0\dagger \qquad (p).$$

5. *In order that* $\mathrm{L}\alpha$ *shall not exist it is necessary and sufficient that there exist* e_0 *and a system* $(p_p|p)$ *such that*

$$p_p\mathrm{R}p,\ |\alpha(p) - \alpha(p_p)| > e_0\dagger \qquad (p).$$

Theorems 4 and 5 follow from 1 and 3 respectively.

The following theorems involve sequences $\{p_n\}$ and a property: *monotone* (R); of sequences and two binary relations R_0, R on the class of sequences. A sequence $\{p_n\}$ is *monotone* (R) in case $p_{n+1}\mathrm{R}p_n$ (n). The notation $\{p'_n\}\mathrm{R}_0\{p''_n\}$ means that $p'_n\mathrm{R}p''_n$ (n). The notation $\{p'_n\}\mathrm{R}\{p''_n\}$ means that there exists a system $(n_n|n)$ such that $p'_{n_n}\mathrm{R}p''_n$ (n). Plainly if the relation R_0 holds for two sequences so does the relation R.

* The ordinary properties of $\mathrm{L}_{n\to\infty}$ are here assumed known.

† If α is multiply valued, this inequality is understood to hold for at least one mode of determining the functional values involved.

6. *If* $\mathrm{L}\alpha = a^*$, *there exists a sequence* $\{p_n^0\}$ *monotone* (R) *such that* $\mathrm{L}_n\alpha(p_n) = a^*$ *for every sequence* $\{p_n\}$ *such that* $\{p_n\}\mathrm{R}_0\{p_n^0\}$, *and also for every sequence* $\{p_n\}$ *monotone* (R) *such that* $\{p_n\}\mathrm{R}\{p_n^0\}$.

Take $\{p_n^0\}$ such that $p_{n+1}^0\mathrm{R}p_n^0$ (n) and

$$|a^* - \alpha(p)| \leqq \frac{1}{n} \qquad (p\mathrm{R}p_n^0) \qquad (n)$$

or

$$\sigma\alpha(p) \geqq n \qquad (p\mathrm{R}p_n^0) \qquad (n),$$

according as a^* is finite or equals $\sigma\infty$, $\sigma = \pm$.

7. $\mathrm{L}\alpha = a^*$ *if there exists a sequence* $\{p_n^0\}$ *such that* $\mathrm{L}_n\alpha(p_n) = a^*$ *for every sequence* $\{p_n\}$ *monotone* (R) *such that* $\{p_n\}\mathrm{R}_0\{p_n^0\}$.

We prove the equivalent contrapositive theorem:

7′. *If it is untrue that* $\mathrm{L}\alpha = a^*$, *then for every sequence* $\{p_n\}$ *there exists a sequence* $\{p_n^0\}$ *monotone* (R) *such that* $\{p_n^0\}\mathrm{R}_0\{p_n\}$ *and it is untrue that* $\mathrm{L}_n\alpha(p_n^0) = a^*$.

We have given e_0 and $(p_p|p)$ such that $p_p\mathrm{R}p$ (p) and (for at least one determination of $\alpha(p_p)$)

$$|a^* - \alpha(p_p)| > e_0 \qquad (p)$$

or

$$\sigma\alpha(p_p) < e_0 \qquad (p)$$

according as a^* is finite or equals $\sigma\infty$, $\sigma = \pm$. Hence from the given sequence $\{p_n\}$ an effective sequence $\{p_n^0\}$ is obtained by recursion in the form $\{p_{p'_n}\}$ on taking $p'_1 = p_1$ and for $n > 1$ $p'_n\mathrm{R}(p_n, p_{n-1}^0)$.

8. *The following conditions on* a^* *and the function* α *are equivalent*:

(*A*) $\mathrm{L}\alpha = a^*$;

(*B*) *There exists a sequence* $\{p_n^0\}$ *such that* $\mathrm{L}_n\alpha(p_n) = a^*$ *for every sequence* $\{p_n\}$ *such that* $\{p_n\}\mathrm{R}_0\{p_n^0\}$;

(*C*) *There exists a sequence* $\{p_n^0\}$ *such that* $\mathrm{L}_n\alpha(p_n) = a^*$ *for every sequence* $\{p_n\}$ *monotone* (R) *such that* $\{p_n\}\mathrm{R}_0\{p_n^0\}$;

(*D*) *There exists a sequence* $\{p_n^0\}$ *such that* $\mathrm{L}_n\alpha(p_n) = a^*$ *for every sequence* $\{p_n\}$ *monotone* (R) *such that* $\{p_n\}\mathrm{R}\{p_n^0\}$.

This follows from 6 and 7. For by 6 *A* implies *B* and *D*; *B* implies *C*; *D* implies *C*; and by 7 *C* implies *A*.

9. *In order that* $\mathrm{L}\alpha$ *shall exist it is necessary and sufficient that there exist a sequence* $\{p_n^0\}$ *such that* $\mathrm{L}_n[\alpha(p_n^0) - \alpha(p_n)] = 0$ *for every sequence* $\{p_n\}$ *such that* $\{p_n\}\mathrm{R}_0\{p_n^0\}$.

The necessity follows from 8. The sufficiency is equivalent to

9′. *If* $\mathrm{L}\alpha$ *does not exist, then for every sequence* $\{p_n\}$ *there exists a sequence* $\{p_n^0\}$ *such that* $\{p_n^0\}\mathrm{R}_0\{p_n\}$ *and it is untrue that* $\mathrm{L}_n[\alpha(p_n) - \alpha(p_n^0)] = 0$.

In proof of 9′ we have given a sequence $\{p_n\}$ and by 5 a number e_0 and a system $(p_p|p)$ such that for every p $p_p\text{R}p$ and $|\alpha(p) - \alpha(p_p)| > e_0$.* Then, taking the sequence $\{p_{p_n}\}$ as sequence $\{p_n^0\}$, we have for every n $p_n^0\text{R}p_n$ and $|\alpha(p_n) - \alpha(p_n^0)| > e_0$,† and accordingly, as stated, $\{p_n^0\}\text{R}_0\{p_n\}$ and the untruth of $\text{L}_n[\alpha(p_n) - \alpha(p_n^0)] = 0$.

§ 3. Some Modes of Convergence.

The non-existence of $\text{L}\alpha$ implies by § 2, 5 the existence of a sequence‡ $\{p_n\}$ monotone (R) such that $\text{L}_n\alpha(p_n)$ does not converge, and for every sequence $\{p_n^0\}$ the existence of a sequence§ $\{p_n\}$ monotone (R) such that $\{p_n\}\text{R}\{p_n^0\}$ and $\Sigma_n|\alpha(p_{n+1}) - \alpha(p_n)| = \infty$. Hence we are led to formulate the following *modes of existence* (or *convergence*) of $\text{L}\alpha$.

$\text{L}\alpha$ *exists absolutely* in case there exists a sequence $\{p_n^0\}$ such that $\Sigma_n|\alpha(p_{n+1}) - \alpha(p_n)| < \infty$ for every sequence $\{p_n\}$ monotone (R) such that $\{p_n\}\text{R}\{p_n^0\}$.

$\text{L}\alpha$ *exists unconditionally* in case $\text{L}_n\alpha(p_n)$ converges finitely (or what is equivalent, in case $\Sigma_n[\alpha(p_{n+1}) - \alpha(p_n)]$ converges finitely) for every sequence $\{p_n\}$ monotone (R).

$\text{L}\alpha$ *exists absolutely-unconditionally* in case $\Sigma_n|\alpha(p_{n+1}) - \alpha(p_n)| < \infty$ for every sequence $\{p_n\}$ monotone (R).

$|\text{L}|\alpha = a$, *the absolute limit of* α *is* a, in case a is the least upper bound of $|\alpha(p_1)| + \Sigma_n|\alpha(p_{n+1}) - \alpha(p_n)|$ for all sequences $\{p_n\}$ monotone (R). Here it would be simpler and equivalent to consider finite (instead of infinite) monotone sequences $\{p_n\}$.

By the initial remark, $\text{L}\alpha$ exists if $\text{L}\alpha$ exists absolutely or unconditionally. Evidently $\text{L}\alpha$ exists absolutely and unconditionally if $\text{L}\alpha$ exists absolutely-unconditionally. Finally, if $|\text{L}|\alpha$ exists, then $\text{L}\alpha$ exists absolutely-unconditionally and clearly $|\text{L}|\alpha \geqq |\text{L}\alpha|$.

That $\text{L}\alpha$ *exist unconditionally it is necessary and sufficient that* $\text{L}\alpha(\mathfrak{P}_0)$ *exist for every subclass* $\mathfrak{P}_0$ *of* $\mathfrak{P}$ *such that* R *as on* $\mathfrak{P}_0\mathfrak{P}_0$ *has the property C.* The condition is necessary, since $\text{L}\alpha$, and *a fortiori* $\text{L}\alpha(\mathfrak{P}_0)$, exists unconditionally, and therefore $\text{L}\alpha(\mathfrak{P}_0)$ exists. To prove it sufficient, that is, that $\text{L}_n\alpha(p_n)$ exists for every $\{p_n\}$ monotone (R), take $\mathfrak{P}_0 = [p_n|n]$. Then there exists a system $(n_e|e)$ such that $|\text{L}\alpha(\mathfrak{P}_0) - \alpha(p_n)| \leqq e$ for every $p_n\text{R}p_{n_e}$, in particular, for every p_n such that $n \geqq n_e + 1$. Hence $\text{L}_n\alpha(p_n)$ exists.

It is readily seen from the second paragraph of § 3 that if $\text{L}\alpha$ exists then as to the existence of $\text{L}\alpha$ absolutely, $\text{L}\alpha$ unconditionally, $\text{L}\alpha$ absolutely-

* For at least one determination of $\alpha(p)$ and $\alpha(p_p)$.

† For at least one determination of $\alpha(p_n)$, $\alpha(p_n^0)$.

‡ Take p_1 at random and for every n $p_{n+1} = p_{p_n}$.

§ Take p_1 at random, $p_2\text{R}(p_1, p_1^0, p_2^0)$, and for every n $p_{2n+1} = p_{p}$, $p_{2n+2}\text{R}(p_{2n+1}, p_{2n+1}^0, p_{2n+2}^0)$.

unconditionally, $|\mathrm{L}|\alpha$, only the six situations can occur which are indicated in the table below.

	$\mathrm{L}\alpha$	$\mathrm{L}\alpha$ abs.	$\mathrm{L}\alpha$ unc.	$\mathrm{L}\alpha$ abs.-unc.	$\|\mathrm{L}\|\alpha$
(I)	+	+	+	+	+
(II)	+	+	+	+	−
(III)	+	+	+	−	−
(IV)	+	+	−	−	−
(V)	+	−	+	−	−
(VI)	+	−	−	−	−

Here a + sign indicates the existence of the concept at the head of the column, a − sign its non-existence. That these six situations actually occur is shown by the following examples (I) $\cdots$ (VI).

Let $A = a_1 + a_2 + \cdots = \mathrm{L}_n A_n$ be an absolutely convergent series. Let $C = c_1 + c_2 + \cdots = \mathrm{L}_n C_n$ be a conditionally convergent series. Denote by $\mathfrak{P}^{\mathrm{III}}$ the class $[n]$ of positive integers n.

(I) $\mathfrak{P} = \mathfrak{P}^{\mathrm{III}}$. $n_1 \mathrm{R} n_2$ in case $n_1 \geqq n_2$. $\alpha(n) = A - A_n$ (n).

(II) $\mathfrak{P} = \mathfrak{P}^{\mathrm{III}} + \infty$. $n_1 \mathrm{R} n_2$ in case $n_1 \doteq n_2$; $\infty \mathrm{R} \infty$; $\infty \mathrm{R} n$ (n). $\alpha(n) = n$ (n); $\alpha(\infty) = 0$.

(III) $\mathfrak{P} = \mathfrak{P}^{\mathrm{III}}$. $n_1 \mathrm{R} n_2$ in case either n_1 odd n_2 even or $n_1 \equiv n_2 \pmod 2$ with $n_1 \geqq n_2$. $\alpha(n) = C - C_{(n+1)/2}$ (n odd), $A - A_{n/2}$ (n even).

(IV) The same as (III) except that $\alpha(n) = n$ (n odd), $A - A_{n/2}$ (n even).

(V) $\mathfrak{P} = \mathfrak{P}^{\mathrm{III}}$. $n_1 \mathrm{R} n_2$ in case $n_1 \geqq n_2$. $\alpha(n) = C - C_n$ (n).

(VI) The same as (III) except that $\alpha(n) = n$ (n odd), $C - C_{n/2}$ (n even)

§ 4. Quasi-Limits. Upper and Lower Limits.

***L**α = a*, a quasi-limit of α is a*,* in case there exists a system $(p_{ep}|ep)$ such that $p_{ep} \mathrm{R} p$ (ep) and

$$|a^* - \alpha(p_{ep})| \geqq e \quad (ep) \qquad \text{or} \qquad \sigma\alpha(p_{ep}) \geqq e \quad (ep),$$

according as a^* is finite or is $\sigma\infty$, $\sigma = \pm$.

***L**₀α = a*, a weak quasi-limit of α is a*,* in case there exists a system $(p_e|e)$ such that

$$|a^* - \alpha(p_e)| \leqq e \quad (e) \qquad \text{or} \qquad \sigma\alpha(p_e) \geqq e \quad (e),$$

according as a^* is finite or is $\sigma\infty$, $\sigma = \pm$.

If $\mathfrak{P}_0$ is a subclass of $\mathfrak{P}$ such that $\mathfrak{P}$ as on $\mathfrak{P}_0\mathfrak{P}_0$ has the properties TC, the notations $\boldsymbol{L}\alpha(\mathfrak{P}_0)$, $\boldsymbol{L}_0\alpha(\mathfrak{P}_0)$ have obvious meanings. For two subclasses $\mathfrak{P}_1\mathfrak{P}_2$ of $\mathfrak{P}$ denote by $\mathfrak{P}_1 \mathrm{R} \mathfrak{P}_2$ the condition that for every p_2 of $\mathfrak{P}_2$ there is a p_1 of $\mathfrak{P}_1$ for which $p_1 \mathrm{R} p_2$. Hence if $\mathfrak{P}_0 \mathrm{R} \mathfrak{P}$ R as on $\mathfrak{P}_0\mathfrak{P}_0$ has the properties TC.

1. *If* $\boldsymbol{L}\alpha = a^*$, *then* $\boldsymbol{L}_0\alpha = a^*$.

2. *In order that* $\mathrm{L}\alpha = a^*$, *it is necessary and sufficient* that $\boldsymbol{L}\alpha(\mathfrak{P}_0) = a^*$ *uniquely for every* $\mathfrak{P}_0 \mathrm{R} \mathfrak{P}$.

The condition is *necessary*. Since $L\alpha = a^*$ and $\mathfrak{P}_0 R \mathfrak{P}$ there is a system $(p_e|e)_0$† such that

$$|a^* - \alpha(p)| \leqq e \quad (pRp_e) \qquad \text{or} \qquad \sigma\alpha(p) \geqq e \quad (pRp_e),$$

according as a^* is finite or is $\sigma\infty$, $\sigma = \pm$. Then by means of a system $(p_{ep}|ep)_0$ such that $p_{ep}R(p, p_e)$ $(ep)_0$, it is clear that $\boldsymbol{L}\alpha(\mathfrak{P}_0) = a^*$. It remains to show that $\boldsymbol{L}\alpha(\mathfrak{P}_0) = a_1^*$ implies $a_1^* = a^*$. From this hypothesis there is a system $(p_{1ep}|ep)_0$ such that we have for every e

$$|a_1^* - \alpha(p_{1ep_e})| \leqq \frac{e}{3} \qquad \text{or} \qquad \sigma_1\alpha(p_{1ep_e}) \geqq \frac{e}{3};$$

$$|a^* \ - \alpha(p_{1ep_e})| \leqq e \qquad \text{or} \qquad \sigma\alpha(p_{1ep_e}) \geqq e$$

(according to the values of a_1^*; a^*), whence the conclusion $a_1^* = a^*$ follows readily.

The condition is *sufficient*. The proof is indirect. The untruth of $L\alpha = a^*$ implies the existence of a positive number e_0 and a system $(p_p|p)$ such that

$$p_pRp \quad (p), \qquad |\alpha(p_p) - a^*| > e_0 \quad (p) \qquad \text{or} \qquad \sigma\alpha(p_p) < e_0 \quad (p),$$

according to the value of a^*. Since $\boldsymbol{L}\alpha = a^*$, $\mathfrak{P}$ being a $\mathfrak{P}_0R\mathfrak{P}$, there is a system $(p_{ep}|ep)$ such that

$$p_{ep}Rp \quad (ep), \qquad |\alpha(p_{ep}) - a^*| \leqq e \quad (ep) \qquad \text{or} \qquad \sigma\alpha(p_{ep}) \geqq e \quad (ep),$$

according to the value of a^*. The class

$$\mathfrak{P}_0 \equiv [p_0] \equiv [p'_{ep} \equiv p_{p_{ep}}|ep]$$

is a class $\mathfrak{P}_0R\mathfrak{P}$. Hence by hypothesis $\boldsymbol{L}\alpha(\mathfrak{P}_0) = a^*$, whereas evidently

$$|\alpha(\mathfrak{P}_0) - a^*| > e_0 \qquad \text{or} \qquad \sigma\alpha(\mathfrak{P}_0) < e_0.$$

This is the desired contradiction. Hence $L\alpha = a^*$, as stated.

We assume α to be real- and single-valued in the remainder of § 4.

Associated with α are two functions $\overline{\alpha}$; $\underline{\alpha}$ (read: α *upper*; α *lower*) on $\mathfrak{P}$ to $\mathfrak{A}^*$, with

$$\overline{\alpha}(p) \equiv \overline{B}_{p_1|p_1Rp}\, \alpha(p_1); \qquad \underline{\alpha}(p) \equiv \underline{B}_{p_1|p_1Rp}\, \alpha(p_1),$$

that is, $\overline{\alpha}(p)$ is the (least) upper bound of $\alpha(p_1)$ for all p_1Rp; $\underline{\alpha}(p)$ is the (greatest) lower bound of $\alpha(p_1)$ for all p_1Rp. Plainly $\overline{\alpha}$; $\underline{\alpha}$ are monotone functions decreasing; increasing, and for every p $\underline{\alpha}(p) \leqq \overline{\alpha}(p)$.

The lower bound $\underline{B}\overline{\alpha} \equiv \underline{B}_p\overline{\alpha}(p)$ of the function $\overline{\alpha}$ is *the upper limit* $\overline{L}\alpha$ of the function α. Similarly the upper bound $\overline{B}\underline{\alpha}$ of $\underline{\alpha}$ is *the lower limit* $\underline{L}\alpha$ of α. It is readily seen that $\underline{L}\alpha \leqq \overline{L}\alpha$.

3. $L\overline{\alpha} = \overline{L}\alpha$; $L\underline{\alpha} = \underline{L}\alpha$.

† In this proof the suffix 0 indicates that the elements p_e, p, p_{ep}, etc., involved belong to $\mathfrak{P}_0$.

This follows from § 1, 7.

4_1. *In order that* $\overline{\mathrm{L}}\alpha = a$, *it is necessary and sufficient that there exist systems* $(p_e | e)$, $(p_{ep} | ep)$ *such that*

$$(1) \qquad \alpha(p) \leqq a + e \quad (p\mathrm{R}p_e) \quad (e),$$

$$(2) \qquad p_{ep}\mathrm{R}p, \qquad \alpha(p_{ep}) \geqq a - e \quad (ep).$$

It is *necessary*. For there are systems $(p_e|e)$, $(p_{ep}|ep)$ such that $a \leqq \overline{\alpha}(p_e) \leqq a + e$ (e), $p_{ep}\mathrm{R}p$ (ep), $\overline{\alpha}(p) - \alpha(p_{ep}) \leqq e$ (ep). These systems satisfy (1), (2). For $\alpha(p) \leqq \overline{\alpha}(p_e) \leqq a + e$ $(p\mathrm{R}p_e)$ (e), and $a \leqq \overline{\alpha}(p) \leqq \alpha(p_{ep}) + e$ (ep).

It is *sufficient*. For by (1) $\overline{\alpha}(p_e) \leqq a + e$ (e); hence $\overline{\mathrm{L}}\alpha \leqq (\alpha p_e) \leqq a + e$ (e), and accordingly $\overline{\mathrm{L}}\alpha \leqq a$; and by (2) $\overline{\alpha}(p) \geqq \alpha(p_{ep}) \geqq a - e$ (ep); hence $\overline{\alpha}(p) \geqq a$ (p), and accordingly $\overline{\mathrm{L}}\alpha \geqq a$. Hence $\overline{\mathrm{L}}\alpha = a$, as stated.

4_2. *In order that* $\underline{\mathrm{L}}\alpha = a$, *it is necessary and sufficient that there exist systems* $(p_e | e)$, $(p_{ep} | ep)$ *such that*

$$(1) \qquad \alpha(p) \geqq a - e \quad (p\mathrm{R}p_e) \quad (e),$$

$$(2) \qquad p_{ep}\mathrm{R}p, \qquad \alpha(p_{ep}) \leqq a + e \quad (ep).$$

5. $\mathrm{L}\alpha = a^*$ *is equivalent to* $\overline{\mathrm{L}}\alpha = \underline{\mathrm{L}}\alpha = a^*$.

6. $\boldsymbol{L}\alpha = \overline{\mathrm{L}}\alpha$, $\boldsymbol{L}\alpha = \underline{\mathrm{L}}\alpha$.

7. $\underline{\mathrm{L}}\alpha \leqq \boldsymbol{L}\alpha \leqq \overline{\mathrm{L}}\alpha$ *for every* $\boldsymbol{L}\alpha$ *finite or infinite.*

Theorems 5, 6, 7 follow readily* from the definitions and theorems 3, 4.

8. $\mathrm{L}\alpha = a^*$ *is equivalent to* $\boldsymbol{L}\alpha = a^*$ *uniquely.*

This theorem follows from 5, 6, 7.

§ 5. Limits as to Norm.

As basis for § 5 we take the system

$$\Sigma_1 = (\mathfrak{A};\ \mathfrak{P};\ \mathrm{R};\ \nu),$$

where $\mathfrak{A}$, $\mathfrak{P}$, R have the same meanings as before and are subject to the same postulates, and where ν, *the norm,* is a real-valued function on $\mathfrak{P}$ to $\mathfrak{A}$ which is monotone decreasing (R) and such that $\underline{B}\nu = 0$.

$\mathrm{L}_\nu\alpha = a^*$, *the limit of* α *is* a^* *as to the norm* ν, provided there exists a system $(d_e|e)$ such that for every p such that $\nu(p) \leqq d_e$

$$|a^* - \alpha(p)| \leqq e \qquad \text{or} \qquad \sigma\alpha(p) \geqq e,$$

according as a^* is finite or is $\sigma\infty$, $\sigma = \pm$.

Associated with ν is a relation R' (having the properties T, C of the relation R) defined as follows: $p_1\mathrm{R}'p_2$ if and only if $\nu(p_1) \leqq \nu(p_2)$. It is

* E.g., if $\underline{L}\alpha = a$, from the systems $(pe|e)(p_{ep}|{}_ep)$ of 4_2 we obtain a system $(\overset{\circ}{p}_{ep}|ep)$ effective for the proof of 6_2: $\boldsymbol{L}\alpha = \underline{\mathrm{L}}\alpha = a$, as follows: For every ep take $p'_{ep}\mathrm{R}(p, p_e)$ and $\overset{\circ}{p}_{ep} \equiv p_{ep'_{ep}}$.

easily shown that L_ν and $L_{R'}$ are equivalent: Whenever either $L_\nu\alpha$ or $L_{R'}\alpha$ exists finite or infinite, the other does and the two limits are equal. Hence the preceding theorems may be applied to L_ν. In particular L_ν is unique.

We are concerned in the sequel with certain relations between L_ν and L_R.

1. *If* $L_\nu\alpha = a^*$, *then* $L_R\alpha = a^*$.

Assume a^* finite. Then there exist systems $(d_e|e)$, $(p_e|e)$ such that

$$|a^* - \alpha(p)| \leqq e \qquad (p \text{ with } \nu(p) \leqq d_e) \quad (e)$$

and $\nu(p_e) \leqq d_e$ (e). Then $|a^* - \alpha(p)| \leqq e$ $(p\text{R}p_e)$, as stated, since $\nu(p) \leqq \nu(p_e) \leqq d_e$ $(p\text{R}p_e)$. The proof is similar for the case a^* infinite.

2. *If* $L_R\alpha$ *exists, then in order that* $L_\nu\alpha$ *shall exist and equal* $L_R\alpha$, *it is necessary and sufficient that there exist systems* $(d_e|e)$, $(p_{ep_1p}|ep_1p)$ *such that*

$$p_{ep_1p}\text{R}p_1, \quad |\alpha(p_{ep_1p}) - \alpha(p)| \leqq e \quad (e,\, p_1,\, p \text{ with } \nu(p) \leqq d_e).$$

The condition is *necessary.* Take systems $(d_e|e)$, $(p_e|e)$, $(p_{ep_1p}|ep_1p)$ such that

$$|\alpha(p) - L_\nu\alpha| \leqq \frac{e}{2} \quad (p \text{ with } \nu(p) \leqq d_e) \quad (e),$$
$$\nu(p_e) \leqq d_e \quad (e),$$
$$p_{ep_1p}\text{R}p_1, \qquad p_{ep_1p}\text{R}p_e \quad (ep_1p).$$

(The p_{ep_1p} may clearly be taken as the same for every p.) Then for every p_1 and e and every p with $\nu(p) \leqq d_e$, we have

$$p_{ep_1p}\text{R}p_1, \quad |\alpha(p) - \alpha(p_{ep_1p})| \leqq |\alpha(p) - L_\nu\alpha| + |L_\nu\alpha - \alpha(p_{ep_1p})| \leqq e,$$

as stated.

The condition is *sufficient.* Take $(p_e|e)$ such that $|\alpha(p) - L_R\alpha| \leqq e/2$ $(p\text{R}p_e)$. Then for every e and p with $\nu(p) \leqq d_{e/2}$, we have

$$|\alpha(p) - L_R\alpha| \leqq |\alpha(p) - \alpha(p_{(e/2)p_ep})| + |\alpha(p_{(e/2)p_ep}) - L_R\alpha| \leqq e,$$

as stated.

All of the classical limits are limits as to a norm, and as such are instances of our general limit. We consider, as examples, the Riemann and Lebesgue integrals.

Let f denote a function of the variable x, x ranging over I: $a \leqq x \leqq b$. Take $\mathfrak{P}$ to be the class of all partitions p of I; a partition of I is a set I_1, $\cdots$, I_n of intervals non-overlapping (except for end points) such that $I = I_1 + \cdots + I_n$. We define $\nu(p)$ for a partition $p = I_1, \cdots, I_n$ as the length of the longest I_k $(k = 1, \cdots, n)$. We say a partition $p' = I'_1$, $\cdots$, $I'_{n'}$ is in the R relation to a partition $p'' = I''_1, \cdots, I''_{n''}$ if p' is a repartition of p'', that is, if every I'_k lies entirely in some I''_k (except possibly for end points). Every function f gives rise to an associated (multiply valued) function α on $\mathfrak{P}$ to $\mathfrak{A}$: $\alpha(p) \equiv \alpha(I_1, \cdots, I_n) \equiv \sum_k^{1n} f(x_k) I_k$, where I_k denotes the length (or measure) of I_k and x_k is any point of I_k. The ordinary Riemann definition of integration for a bounded function f is, except for form, as follows: If the function f is such that for the associated function α, $\mathrm{L}_\nu \alpha$ exists, then $\mathrm{L}_\nu \alpha$ is called the (Riemann) integral of f from a to b and is denoted by $\int_a^b f(x)dx$. But it follows from 1 and 2 above that $\mathrm{L}_{\mathrm{R}}\alpha$ used in place of $\mathrm{L}_\nu \alpha$ in the definition just given would yield a second and equivalent definition.

This remark is important in that it leads to a simple and natural definition of the Lebesgue integral $\int_a^b f(x)dx$ as a limit. To secure such a definition from the second definition it is only necessary to define partition differently, a partition now being a set of non-overlapping measurable sets $I_1, \cdots, I_n$ such that $I = I_1 + \cdots + I_n$. The junior author hopes soon to publish a theory of integration from this point of view which has been in his possession since 1917.

§ 6. Types of Uniform Convergence as to a General Parameter.

The *fundamental system* for § 6 is

$$\Sigma_2 \equiv (\mathfrak{A};\ \mathfrak{P};\ \mathrm{R}^{\text{on } \mathfrak{P}\mathfrak{P}.TC};\ \mathfrak{Q}),$$

that is, the fundamental system of § 1 with the adjunction of a *general class* $\mathfrak{Q} \equiv [q]$ of elements q.

We consider functions $\alpha \equiv (\alpha(p)\,|\,p)$ on $\mathfrak{P}$ to $\mathfrak{A}$; $\beta \equiv (\beta(q)\,|\,q)$ on $\mathfrak{Q}$ to $\mathfrak{A}$; $\varphi \equiv (\varphi(pq)\,|\,pq)$ on $\mathfrak{P}\mathfrak{Q}$ to $\mathfrak{A}$. Thus a function φ is a function of the variables pq which range independently over the (conceptually, but not necessarily actually, distinct) classes $\mathfrak{P}\mathfrak{Q}$; we denote by $\varphi(\diamond q) \equiv (\varphi(pq)\,|\,p)$, $\varphi(p\diamond) \equiv (\varphi(pq)\,|\,q)$ the functions α on $\mathfrak{P}$ to $\mathfrak{A}$, β on $\mathfrak{Q}$ to $\mathfrak{A}$ obtained from φ by fixing the respective arguments q, p. With respect to the limits now to be defined the argument q plays the rôle of a parameter.

$\mathrm{L}\varphi = \beta$ ($\mathfrak{Q}$; unif.), *the limit of* φ *is* β *over* $\mathfrak{Q}$ *uniformly,* in case there exists a system $(p_e\,|\,e)$ such that

$$|\varphi(p\diamond) - \beta| \leqq e \quad (p\mathrm{R}p_e) \quad (e).$$

$\mathrm{L}\varphi = \beta$ ($\mathfrak{Q}$; quasi-unif.), *the limit of* φ *is* β *over* $\mathfrak{Q}$ *quasi-uniformly,* in case there exists a system $(p_{eq}\,|\,eq)$ such that

(1) the set $[p_{eq}\,|\,q]$ is finite (e),
(2) $|\varphi(pq) - \beta(q)| \leqq e \quad (p\mathrm{R}p_{eq}) \quad (eq)$.

$\mathrm{L}\varphi = \beta$ ($\mathfrak{Q}$; semi-unif.), *the limit of* φ *is* β *over* $\mathfrak{Q}$ *semi-uniformly,* in case there exists a system $(p_{eq}\,|\,eq)$ such that

(1) the set $[p_{eq}\,|\,q]$ is (finitely or infinitely) denumerable (e),
(2) $|\varphi(pq) - \beta(q)| \leqq e \quad (p\mathrm{R}p_{eq}) \quad (eq)$.

$\boldsymbol{L}\varphi = \beta$ ($\mathfrak{Q}$; unif.), *a quasi-limit of* φ *is* β *over* $\mathfrak{Q}$ *uniformly,* in case there exists a system $(p_{ep}|ep)$ such that

$$p_{ep}\text{R}p, \quad |\varphi(p_{ep}\diamond) - \beta| \leqq e \quad (ep).$$

$\boldsymbol{L}\varphi = \beta$ ($\mathfrak{Q}$; quasi-unif.), *a quasi-limit of* φ *is* β *over* $\mathfrak{Q}$ *quasi-uniformly,* in case there exists a system $(p_{epq}|epq)$ such that

(1) the set $[p_{epq}|q]$ is finite (ep),
(2) $p_{epq}\text{R}p, \quad |\varphi(p_{epq}q) - \beta(q)| \leqq e \quad (epq)$.

$\boldsymbol{L}\varphi = \beta$ ($\mathfrak{Q}$; semi-unif.), *a quasi-limit of* φ *is* β *over* $\mathfrak{Q}$ *semi-uniformly,* in case there exists a system $(p_{epq}|epq)$ such that

(1) the set $[p_{epq}|q]$ is denumerable (ep),
(2) $p_{epq}\text{R}p, \quad |\varphi(p_{epq}q) - \beta(q)| \leqq e \quad (epq)$.

From the definitions of the three quasi-limits $\boldsymbol{L}$ we obtain definitions of the corresponding *weak quasi-limits* $\boldsymbol{L}_0$ by omitting the subscript p and the conditions involving R.

These types of uniform convergence as to a parameter are for use in §§ 7, 9. Obviously the quasi-uniform convergence L and the uniform convergence L are equivalent, and the quasi-uniform convergences L, $\boldsymbol{L}$, $\boldsymbol{L}_0$ imply the semi-uniform convergences L, $\boldsymbol{L}$, $\boldsymbol{L}_0$ respectively.

§ 7. Double Limits.

From two systems
$\Sigma' \equiv (\mathfrak{A};\ \mathfrak{P}' \equiv [p'];\ \text{R}'^{\text{on}\ \mathfrak{P}'\mathfrak{P}'.TC}),\ \Sigma'' \equiv (\mathfrak{A};\ \mathfrak{P}'' \equiv [p''];\ \text{R}''^{\text{on}\ \mathfrak{P}''\mathfrak{P}''.TC})$
of the type studied in §§ 1–4 we form the *composite system*

$$\Sigma \equiv (\mathfrak{A};\ \mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}'';\ \text{R} \equiv \text{R}'\text{R}''^{\text{on}\ \mathfrak{P}\mathfrak{P}.TC})$$

of the same type. Here $\mathfrak{P}'\ \mathfrak{P}''$ are two general classes conceptually (but not necessarily actually) distinct; $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$ is the product or composite class $[p] \equiv [p'p'']$ of all composite elements $p \equiv p'p''$; and $\text{R} \equiv \text{R}'\text{R}''$ is the composite relation R on $\mathfrak{P}\mathfrak{P}$: $p_1\text{R}p_2 \equiv p_1'p_1''\text{R}p_2'p_2''$ in case $p_1'\text{R}'p_2'$ and $p_1''\text{R}''p_2''$.

We denote limits, quasi-limits, weak quasi-limits as to R′; R″; R of functions α'; α''; α on $\mathfrak{P}'$; $\mathfrak{P}''$; $\mathfrak{P}$ to $\mathfrak{A}$ by L′, $\boldsymbol{L}'$, $\boldsymbol{L}_0'$; L″, $\boldsymbol{L}''$, $\boldsymbol{L}_0''$; L, $\boldsymbol{L}$, $\boldsymbol{L}_0$, respectively. For a function α on $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$ to $\mathfrak{A}$ L′α is said to exist in case for every p'' L′$\alpha(\diamond p'')$ exists, and in this case L′α denotes the function $(\text{L}'\alpha(\diamond p'')|p'')$ on $\mathfrak{P}''$ to $\mathfrak{A}$; etc.

The *iterated double limits* L′L″, L″L′ and *quasi-limits* $\boldsymbol{L}'$L″, $\boldsymbol{L}_0'$L″, $\boldsymbol{L}''$L′, $\boldsymbol{L}_0''$L′ have evident definitions. The *simultaneous double limit* (L′L″) ≡ (L″L′) and *quasi-limits* $(\boldsymbol{L}'\boldsymbol{L}'') \equiv (\boldsymbol{L}''\boldsymbol{L}')$, $(\boldsymbol{L}_0'\boldsymbol{L}_0'') \equiv (\boldsymbol{L}_0''\boldsymbol{L}_0')$ are defined as the simple limit L and quasi-limits $\boldsymbol{L}$, $\boldsymbol{L}_0$ for the composite system Σ. The *simultaneous double limits* $(\boldsymbol{L}'\mathrm{L}'') \equiv (\mathrm{L}''\boldsymbol{L}')$, $(\boldsymbol{L}_0'\mathrm{L}'') \equiv (\mathrm{L}''\boldsymbol{L}_0')$ are defined as follows:

$(\boldsymbol{L}'\mathrm{L}'')\alpha = a$, in case there exists a system $(p'_{ep'},\ p''_{ep'} \,|\, ep')$ such that

$$p'_{ep'}\mathrm{R}'p', \quad |a - \alpha(p'_{ep'}p'')| \leqq e \quad (p''\mathrm{R}''p''_{ep'})(ep');$$

$(\boldsymbol{L}_0'\mathrm{L}'')\alpha = a$, in case there exists a system $(p'_e,\ p''_e \,|\, e)$ such that

$$|a - \alpha(p'_e p'')| \leqq e \quad (p''\mathrm{R}''p''_e) \quad (e).$$

The limits $(\boldsymbol{L}''\mathrm{L}')$, $(\boldsymbol{L}_0''\mathrm{L}')$ are defined similarly.

1. *If* $\mathrm{L}\alpha$ *and* $\mathrm{L}'\alpha$ *exist, then* $\mathrm{L}''\mathrm{L}'\alpha$ *exists and is equal to* $\mathrm{L}\alpha$.

2. *If* $\mathrm{L}'\alpha$ *and* $\mathrm{L}''\alpha$ *exist, then in order that* $\mathrm{L}\alpha$ *shall exist it is necessary and sufficient that* $\mathrm{L}(\alpha - \mathrm{L}'\alpha) = 0$.

The condition is *necessary.** For by 1 $\mathrm{L}''\mathrm{L}'\alpha$ exists, equal to $\mathrm{L}\alpha$. Hence $\mathrm{L}\mathrm{L}'\alpha$ exists, equal to $\mathrm{L}\alpha$, and

$$\mathrm{L}(\alpha - \mathrm{L}'\alpha) = \mathrm{L}\alpha - \mathrm{L}\mathrm{L}'\alpha = \mathrm{L}\alpha - \mathrm{L}\alpha = 0.$$

The condition is *sufficient.* From the hypotheses we readily secure a system $(p'_e,\ p''_{1e},\ p''_{2e},\ p''_e \,|\, e)$ such that for every e

$$|\alpha(p'p'') - \mathrm{L}'\alpha(\diamond p'')| \leqq \frac{e}{3} \quad (p'\mathrm{R}'_*p'_e,\ p''\mathrm{R}''_*p''_{1e}),$$

$$|\alpha(p'_e p''_1) - \alpha(p'_e p''_2)| \leqq \frac{e}{3} \quad (p''_1\mathrm{R}''p''_{2e},\ p''_2\mathrm{R}''p''_{2e}),$$

$$p''_e\mathrm{R}''p''_{1e}, \quad p''_e\mathrm{R}''p''_{2e}.$$

Then for every e $(p'\mathrm{R}'p'_e,\ p''\mathrm{R}''p''_e)$ implies

$$|\alpha(p'p'') - \alpha(p'_e p''_e)| \leqq |\alpha(p'p'') - \mathrm{L}'\alpha(\diamond p'')| + |\mathrm{L}'\alpha(\diamond p'') - \alpha(p'_e p'')| + |\alpha(p'_e p'') - \alpha(p'_e p''_e)| \leqq e,$$

and hence by § 2, 3 $\mathrm{L}\alpha$ exists, as stated.

3. *If* $\mathrm{L}'\alpha$ *exists* ($\mathfrak{P}''$; unif.) *and* $\mathrm{L}''\alpha$ *exists, then* $\mathrm{L}\alpha$, $\mathrm{L}'\mathrm{L}''\alpha$, $\mathrm{L}''\mathrm{L}'\alpha$ *exist and are equal.*

This theorem follows from 2 and 1, the uniformity implying the sufficient condition of 2.

4_1. *If* $\mathrm{L}''\alpha$ *exists, then in order that* $(\boldsymbol{L}'\mathrm{L}'')\alpha = a$ *it is necessary and sufficient that* $\boldsymbol{L}'\mathrm{L}''\alpha = a$.

It is *necessary.* We readily secure a system $(p'_{ep'},\ p''_{ep'} \,|\, ep')$ such that

$$p'_{ep'}\mathrm{R}'p', \quad |a - \alpha(p'_{ep'}p'')| \leqq \frac{e}{2} \quad (p''\mathrm{R}''_*p''_{ep'}) \quad (ep'),$$

$$|\alpha(p'_{ep'}p''_{ep'}) - \mathrm{L}''\alpha(p'_{ep'}\diamond)| \leqq \frac{e}{2} \quad (ep').$$

* This proof of the necessity was suggested by Professor W. A. Hurwitz of Cornell University.

Then

$$|a - \mathrm{L}''\alpha(p'_{ep'}\diamond)| \leqq |a - \alpha(p'_{ep'}p''_{ep'})| + |\alpha(p'_{ep'}p''_{ep'}) - \mathrm{L}''\alpha(p'_{ep'}\diamond)| \leqq e \quad (ep').$$

It is *sufficient.* We secure a system $(p'_{ep'}, p''_{ep'} | ep')$ such that

$$p'_{ep'}\mathrm{R}'p', \quad |a - \mathrm{L}''\alpha(p'_{ep'}\diamond)| \leqq \frac{e}{2} \qquad (ep'),$$

$$|\mathrm{L}''\alpha(p'_{ep}\diamond) - \alpha(p'_{ep'}p'')| \leqq \frac{e}{2} \qquad (p''\mathrm{R}''p''_{ep'}) \quad (ep').$$

Then

$$|a - \alpha(p'_{ep'}p'')| \leqq |a - \mathrm{L}''\alpha(p'_{ep'}\diamond)| + |\mathrm{L}''\alpha(p'_{ep'}\diamond) - \alpha(p'_{ep'}p'')| \leqq e \quad (p''\mathrm{R}''p''_{ep'}) \quad (ep').$$

4_2. *If* $\mathrm{L}'\alpha$ *exists, then in order that* $(\mathrm{L}'\boldsymbol{L}'')\alpha = a$ *it is necessary and sufficient that* $\boldsymbol{L}''\mathrm{L}'\alpha = a$.

5. *If* $\mathrm{L}'\alpha$, $\mathrm{L}''\alpha$, $\mathrm{L}'\mathrm{L}''\alpha$ *exist, then in order that* $\mathrm{L}''\mathrm{L}'\alpha$ *exist and equal* $\mathrm{L}'\mathrm{L}''\alpha$ *it is necessary and sufficient that* $(\boldsymbol{L}'\mathrm{L}'')(\alpha - \mathrm{L}'\alpha) = 0$.

It is *necessary.* Now $\mathrm{L}''(\alpha - \mathrm{L}'\alpha)$ exists equal to $\mathrm{L}''\alpha - \mathrm{L}''\mathrm{L}'\alpha$, since each of thelatter limits exists. Since by § 4, 2 $\boldsymbol{L}'\mathrm{L}''\alpha$ exists uniquely equal to $\mathrm{L}'\mathrm{L}''\alpha$ and obviously $\boldsymbol{L}'\mathrm{L}''\mathrm{L}'\alpha$ exists uniquely equal to $\mathrm{L}''\mathrm{L}'\alpha$, we have

$$\boldsymbol{L}'\mathrm{L}''(\alpha - \mathrm{L}'\alpha) = \mathrm{L}'\mathrm{L}''\alpha - \mathrm{L}''\mathrm{L}'\alpha = 0.$$

Hence by 4_1, $(\boldsymbol{L}'\mathrm{L}'')(\alpha - \mathrm{L}'\alpha) = 0$.

It is *sufficient.* We secure a system $(p'_{1e}, p'_{2e}, p''_e | e)$ such that for every e

$$|\mathrm{L}'\mathrm{L}''\alpha - \mathrm{L}''\alpha(p'\diamond)| \leqq \frac{e}{3} \quad (p'\mathrm{R}'p'_{1e}),$$

$$p'_{2e}\mathrm{R}'p'_{1e}, \quad |\alpha(p'_{2e}p'') - \mathrm{L}'\alpha(\diamond p'')| \leqq \frac{e}{3} \quad (p''\mathrm{R}''_*p''_e).$$

Then

$$|\mathrm{L}'\mathrm{L}''\alpha - \mathrm{L}'\alpha(\diamond p'')| \leqq |\mathrm{L}'\mathrm{L}''\alpha - \mathrm{L}''\alpha(p'_{2e}\diamond)| + |\mathrm{L}''\alpha(p'_{2e}\diamond) - \alpha(p'_{2e}p'')| + |\alpha(p'_{2e}p'') - \mathrm{L}'\alpha(\diamond p'')| \leqq e \quad (p''\mathrm{R}''p''_2) \quad (e).$$

6. *If* $\mathrm{L}'\alpha$, $\mathrm{L}''\alpha$ *exist, then in order that* $\mathrm{L}'\mathrm{L}''\alpha$ *shall exist it is sufficient that* $(\mathrm{L}'\boldsymbol{L}''_0)(\alpha - \mathrm{L}''\alpha) = 0$.

We secure a system $(p'_e, p''_e | e)$ such that for every e

$$|\alpha(p'p''_e) - \mathrm{L}''\alpha(p'\diamond)| \leqq \frac{e}{3} \quad (p'\mathrm{R}'_*p'_e),$$

$$|\alpha(p'p''_e) - \alpha(p'_e p''_e)| \leqq \frac{e}{3} \quad (p'\mathrm{R}'p'_e).$$

Then

$$|\mathrm{L}''\alpha(p'\diamond) - \mathrm{L}''\alpha(p'_e\diamond)| \leqq |\mathrm{L}''\alpha(p'\diamond) - \alpha(p'p''_e)| + |\alpha(p'p''_e) - \alpha(p'_e p''_e)| + |\alpha(p'_e p''_e) - \mathrm{L}''\alpha(p'_e\diamond)| \leqq e \quad (p'\mathrm{R}'p'_e) \quad (e).$$

7. *If* $L'\alpha$ *and* $L''\alpha$ *exist, then by* 5, 6 *the following four statements are mutually equivalent*:

(1) $L'L''\alpha$ *and* $L''L'\alpha$ *exist and are equal*;
(2) $L'L''\alpha$ *exists and* $(\boldsymbol{L}'L'')(\alpha - L'\alpha) = 0$;
(3) $L''L'\alpha$ *exists and* $(\boldsymbol{L}''L')(\alpha - L''\alpha) = 0$;
(4) $(\boldsymbol{L}'L'')(\alpha - L'\alpha) = 0$, $(L'\boldsymbol{L}_0'')(\alpha - L''\alpha) = 0$.

8. *If* $L'\alpha$, $L''\alpha$, $L'L''\alpha$ *exist, then in order that* $L''L'\alpha$ *shall exist and equal* $L'L''\alpha$ *it is sufficient that*

$$\boldsymbol{L}'(\alpha - L'\alpha) = 0 \quad (\mathfrak{P}''; \text{ quasi-unif.}).$$

We secure systems $(p'_e|e)$, $(p'_{ep''}|ep'')$, $(p''_e|e)$ such that for every e

$$|L''\alpha(p'\diamond) - L'L''\alpha| \leqq \frac{e}{3} \quad (p'\text{R}'p'_e),$$

$$[p'_{ep''}|p''] \text{ is finite},$$

$$p'_{ep''}\text{R}'p'_e, \quad |L'\alpha(\diamond p'') - \alpha(p'_{ep''}p'')| \leqq \frac{e}{3} \quad (p''),$$

$$|\alpha(p'_{ep''}p'') - L''\alpha(p'_{ep''}\diamond)| \leqq \frac{e}{3} \quad (p''\text{R}''p''_e).\dagger$$

Then for every e $p''\text{R}''p''_e$ implies

$$|L'\alpha(\diamond p'') - L'L''\alpha| \leqq |L'\alpha(\diamond p'') - \alpha(p'_{ep''}p'')| + |\alpha(p'_{ep''}p'') - L''\alpha(p'_{ep''}\diamond)| + |L''\alpha(p'_{ep''}\diamond) - L'L''\alpha| \leqq e,$$

and accordingly by § 2, 3 $L''L'\alpha$ exists equal to $L'L''\alpha$, as stated.

§ 8. Lemmas: Revised Formulation of Certain Theorems of Fréchet.*

As foundation for § 8 we have the system

$$\Sigma_3 \equiv (\mathfrak{A};\ \mathfrak{Q};\ \mathfrak{S}^{1.2};\ L^{\text{on } \mathfrak{S} \text{ to } \mathfrak{Q}\cdot 1.2}),$$

where $\mathfrak{Q} \equiv [q]$ is a class of elements q; $\mathfrak{S} \equiv [s]$ is a class of sequences $s \equiv \{q_n\}$ of elements q_n of $\mathfrak{Q}$ to which belong 1) for every q of $\mathfrak{Q}$ the iterative sequence $\dot{q} \equiv \{q_n = q|n\}$ and 2) every subsequence s_0 of a sequence s of $\mathfrak{S}$; L is a single-valued function on $\mathfrak{S}$ to $\mathfrak{Q}$, associating with every sequence $s \equiv \{q_n\}$ of $\mathfrak{S}$ a definite element q, *the limit of* s, denoted by Ls, $L\{q_n\}$ or L_nq_n, for which 1) $L\dot{q} = q\ (q)$ and 2) $Ls_0 = Ls$ for every sequence s of $\mathfrak{S}$ and subsequence s_0 of s.

We denote by Q a non-null subset of the class $\mathfrak{Q}$. For every two sets Q_1 Q_2 the *sum* $Q_1 + Q_2$ and the *product* Q_1Q_2 are respectively the least

† This p''_e exists since $L''\alpha(p'\diamond)$ exists for every p' and the elements $p'_{ep''}$ for fixed e are finite in number.

* Fréchet, Sur quelques points du Calcul Fonctionnel, *Rendiconti . . . di Palermo*, Vol. 22 (1906).

common superset and the greatest common subset of the two sets, while the *difference* $Q_1 - Q_2$ is the set of all elements q of Q_1 but not of Q_2; thus $Q_1 + Q_2$ is a set Q while one (but not both) of the sets Q_1Q_2, $Q_1 - Q_2$ may be the null set.

A set Q is *closed* in case for every s (of $\mathfrak{S}$) in Q the limit Ls is of Q. A set Q is *compact* in case every infinite subset of Q contains an s (of $\mathfrak{S}$) consisting of distinct elements. A set Q_1 is *a region relative to a set* Q_2, in notation, a region (Q_2), in case every sequence s in Q_2 with Ls of Q_1 is ultimately in Q_1, i.e., every element of s after a certain one is an element of Q_1; accordingly, every set Q is a region (Q). A set Q is *covered by* a set [R] of regions (Q) in case every q of Q is of some R of [R]. A set Q is *enclosable* in case there exists a denumerable set [R] of regions (Q) such that 1) Q is covered by [R] and 2) if an element q of Q is an element of a region (Q), say Q_0, then there exists a region R of [R] contained in Q_0 and containing q.

1. *If* $\{Q_n\}$ *is a sequence of closed compact sets* Q_n, *each containing the following, then the product of all the sets* Q_n *contains at least one element* q.

2. *If a closed compact set* Q *is covered by a denumerable set* $[R_n|n]$ *of regions* (Q), *then the set* Q *is covered by some finite subset of the set* $[R_n|n]$.

For* otherwise, in contradiction to 1, $\{Q - (R_1 + \cdots + R_n)\}$ is a sequence $\{Q_n\}$ of closed compact sets, each containing the following, whose product is the null set, since every element q of Q is of some region R_n and hence not of the corresponding set Q_n.

3. *If an enclosable set* Q *is covered by a set* [R] *of regions* (Q), *then it is covered by some denumerable subset of the set* [R].

Denote by $[R_1]$ a denumerable set of regions (Q) covering the set Q in the sense of the enclosability of Q. An element q lies in a region R and accordingly in a region R_1 contained in R. Thus the set Q is covered by a necessarily denumerable subset $[R_2]$ of the set $[R_1]$ each of which is in a region R. Accordingly the set Q is covered by a denumerable subset $[R_3]$ of the set [R], as stated.

From 2, 3 we have the Heine-Borel-Lebesgue theorem:

4. *If a closed compact enclosable set* Q *is covered by a set* [R] *of regions* (Q), *then the set* Q *is covered by some finite subset of the set* [R].

A function β on Q to $\mathfrak{A}$ is *continuous at an element* q of Q, in case $L_n\beta(q_n) = \beta(q)$ for every sequence $\{q_n\}$ in Q with $L_nq_n = q$; and it is *continuous on* Q, in case it is continuous at every q of Q.

§ 9. **Composite Range. Continuity.**

The theorems of this section are with reference to the foundation.

$$\Sigma_4 = (\mathfrak{A};\ \mathfrak{P};\ \mathrm{R}^{\text{on } \mathfrak{P}\mathfrak{P}\cdot TC};\ \mathfrak{Q};\ \mathfrak{S}^{1.2};\ \mathrm{L}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{Q}\cdot 1.2}),$$

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* Compare Hausdorff, *Grundzüge der Mengenlehre*, p. 272.

where the notations have the same meanings as in §§ 1, 6, 8.

We consider a subset Q of $\mathfrak{Q}$ and a function φ on $\mathfrak{P}$Q to $\mathfrak{A}$.

1. *If the function* $\varphi(p\diamond)$ *is continuous on the set* Q *for every* p, *and* $L\varphi(\diamond q)$ *exists for every* q *of* Q, *then in order that* $L\varphi$ *shall be continuous on* Q *it is sufficient that*

$$\boldsymbol{L}(\varphi - L\varphi) = 0 \quad (\text{Q; quasi-unif.}).$$

Consider an element q of Q and a sequence $\{q_n\}$ of Q with $L_n q_n = q$. We are to prove that $L_n L\varphi(\diamond q_n)$ exists and is equal to $L\varphi(\diamond q)$. This is a consequence of § 7, 8. For $L\varphi(\diamond q_n)$ exists for every n; $L_n\varphi(pq_n)$ exists for every p; $LL_n\varphi(\diamond q_n)$, quâ $L\varphi(\diamond q)$, exists, and the sufficient condition stated implies

$$\boldsymbol{L}[\varphi(\diamond q_n) - L\varphi(\diamond q_n)] = 0 \quad ([n];\ \text{quasi-unif.}).$$

2. *If the set* Q *is compact and closed, and the function* $\varphi(p\diamond)$ *is continuous on* Q *for every* p, $L\varphi(\diamond q)$ *exists for every* q *of* Q, *and* $\boldsymbol{L}(\varphi - L\varphi) = 0$ (Q; semi-unif.), *then in order that* $L\varphi$ *shall be continuous on* Q, *it is necessary that* $\boldsymbol{L}(\varphi - L\varphi) = 0$ (Q; quasi-unif.).

There is a set $(p_{1epq} | epq^{\text{ of Q}})$ such that

1) $[p_{1epq} | q^{\text{ of Q}}]$ is denumerable (ep),

2) $p_{1epq}\text{R}p$, $|\varphi(p_{1epq}q) - L\varphi(\diamond q)| \leqq \frac{e}{2} < e \quad (epq^{\text{ of Q}})$.

For every $(epq^{\text{ of Q}})$ let Q_{epq} denote the set of all elements q' of Q for which

$$|\varphi(p_{1epq}q') - L\varphi(\diamond q')| < e.$$

Every set Q_{epq} is a region (Q). Further for every (ep) the set $[Q_{epq} | q^{\text{ of Q}}]$ is a denumerable set of regions (Q) covering Q. Hence by § 8, 2 there is for every (ep) a finite number of them

$$[Q_{epq_1}, Q_{epq_2}, \cdots]$$

which cover Q; for every q of Q set $p_{epq} \equiv p_{1epq_i}$ where i is the smallest integer such that Q_{epq_i} contains q. Then the system $(p_{epq} | epq^{\text{ of Q}})$ is effective in proof of the desired quasi-uniformity.

3. *If the set* Q *is enclosable and the function* $\varphi(p\diamond)$ *is continuous on* Q *for every* p *and* $L\varphi(\diamond q)$ *exists for every* q *of* Q, *then if* $L\alpha$ *is continuous on* Q *it is true that*

$$\boldsymbol{L}(\varphi - L\varphi) = 0 \quad (\text{Q; semi-unif.}).$$

For since $L(\varphi - L\varphi) = L\varphi - LL\varphi = L\varphi - L\varphi = 0$ (Q), we have $\boldsymbol{L}(\varphi - L\varphi) = 0$ (Q); hence there exists a set $(p_{1epq} | epq^{\text{ of Q}})$ such that

$$p_{1epq}\text{R}p, \quad |\varphi(p_{1epq}q) - L\varphi(\diamond q)| \leqq \frac{e}{2} < e \ (epq^{\text{ of Q}}).$$

For every ($epq^{\text{ of Q}}$) let Q_{epq} denote the set of all elements q' of Q for which $|\varphi(p_{1epq}q') - L\varphi(\diamond q')| < e$. Then every Q_{epq} is a region (Q). Further for every (ep) $[Q_{epq}|q^{\text{of Q}}]$ is a class of regions (Q) covering Q. Then by § 8, 3 there is a denumerable subset $[Q_{epq_1}, Q_{epq_2}, \cdots]$ of $[Q_{epq}|q^{\text{ of Q}}]$ covering Q; for every q of Q set $p_{epq} \equiv p_{epq_i}$, where i is the smallest integer such that Q_{epq_i} contains q. Then the system $(p_{epq}|epq^{\text{ of Q}})$ is effective in proof of the desired semi-uniformity.

From 1, 2, 3 we have the theorem of Arzelà:*

4. *If* $\varphi(p\diamond)$ *is continuous on* Q *for every* p *and* $L\varphi(\diamond q)$ *exists for every* q *of* Q, *then in order that* $L\varphi$ *shall be continuous on* Q *it is sufficient that* $\boldsymbol{L}(\varphi - L\varphi) = 0$ (Q; quasi-unif.). *If* Q *is compact, closed and enclosable, then that condition is necessary.*

5. *If* $\varphi(\diamond q)$ *is monotone decreasing for every* q *of* Q, *it is true that*

(1) *if* $\boldsymbol{L}\varphi$ *exists* (Q), *then* $L\varphi$ *exists* (Q);

(2) *if* $\boldsymbol{L}\varphi$ *exists* (Q; semi-unif.), *then* $L\varphi$ *exists* (Q; semi-unif.).

(3) *if* $\boldsymbol{L}\varphi$ *exists* (Q; quasi-unif.), *then* $L\varphi$ *exists* (Q; unif.).

This is easily proved. From 4, 5_3 we have at once the theorem of Dini:

6. *If* Q *is closed, compact and enclosable, and the function* $\varphi(p\diamond)$ *is continuous on* Q *for every* p *and* $\varphi(\diamond q)$ *is monotone decreasing for every* q *of* Q, *and* $\boldsymbol{L}\varphi = 0$, *then* $L\varphi = 0$ (Q; unif.).

THE UNIVERSITY OF CHICAGO,
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* Arzelà, "Sulle Serie di Funzioni," *Rend. di Bologna*, Ser. V, Vol. 8, 1899.